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ON THE MONADIC THEORY OF ω_1 WITHOUT A.C.

BY

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ABSTRACT

Buchi in *Lecture Notes in Mathematics, Decidable Theories II* (1973) by using A.C. characterized the theories $MT[\beta, <]$ for $\beta < \omega_1$ and showed that $MT[\omega_1, <]$ is decidable. We extend Buchi's results to a larger class of models of ZF (without A.C.) by proving the following under ZF only: (1) There is a choice function which chooses a "good" run of an automaton on countable input (Lemma 5.1). It follows that Buchi's results cocerning countable ordinals are provable within ZF. (2) Let U.D. be the assertion that there exists a uniform denumeration of ω_1 (i.e. a function $f: \omega_1 \to \omega_1^\omega$ such that for every $\alpha < \omega_1$, $f(\alpha)$ is a function from ω onto α). We show that U.D. can be stated as a monadic sentence, and therefore ω_1 is characterizable by a sentence. (3) Let F be the filter of the cofinal closed subsets of ω_1. We show that if U.D. holds then $MT[\omega_1, <]$ is recursive in the first order theory of the boolean algebra $P(\omega_1)/F$. (We can effectively translate each monadic sentence Σ to a boolean sentence σ such that $[\omega_1, <] \models \Sigma$ iff $P(\omega_1)/F \models \sigma$). (4) As every complete boolean algebra theory is recursive we have that in every model of ZF + U.D., $MT[\omega_1, <]$ is recursive. All our proofs are within ZF. Buchi's work is often referred to. Following Buchi, the main tool is finite automata. We don't deal with $MT[\omega_1, <]$ for ω_1 which doesn't satisfy U.D.

1. Preliminaries and notation

The monadic second order language has two types of variables: individual variables $(x, y, z, t, \ldots)$ and set variables $(X, Y, Z, \ldots)$. It has relation letters R_i which are applied on individual variables only and the membership relation letter $\in$ between individual variables and set variables (e.g. $x \in X$).

Formulas are built up from atomic ones by the usual logical connectives and the quantifiers $\forall, \exists$, applied on both types of variables.

When we come to define the satisfaction relation between a structure $[D, R_i,$

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$R_2, \ldots]$ and a formula, we have in mind a preset model M of ZF, to which the structure belongs. The individual variables range over the structure domain D, and the set variables range over all the subsets of D in M.

Let $[D, R_1, R_2, \ldots]$ be a structure; by $MT[D, R_1, R_2, \ldots]$ we denote the set of sentences holding in the structure. We limit our discussion to the language having the relations $<$ and $=$ only, and to models satisfying the order axiom "$<$ is a linear order" and the well-ordered axiom $\forall X \exists x \forall y \ (y \in X \rightarrow x \leqq y \wedge x \in X)$. Any such model is well-ordered, and, therefore, isomorphic to an ordinal.

We identify ordinals α with the set of smaller ordinals $\{\beta \mid \beta < \alpha\}$. ω_1 is the first uncountable ordinal. We shall not deal with ordinals above ω_1. For an ordinal α, α' denotes the successor of α. For ordinals α, β, define the intervals $[\alpha, \beta) = \{\gamma \mid \alpha \leqq \gamma < \beta\}$, $[\alpha, \beta] = [\alpha, \beta')$. Let A be a set of ordinals and x an ordinal; we say that A is cofinal in x if $\forall y [y < x \rightarrow \exists t \ (y < t < x \wedge t \in A)$. We sat that A is cofinal if A is cofinal in ω_1. A set A is closed if A contains every ordinal x such that A is cofinal in x (except for ω_1).

We shall write $(\exists^x t) R(t)$ to denote that $\{t \mid R(t)\}$ is cofinal in x. When x appears in such notations it is assumed that x is a limit ordinal. Let $L_M(X)$ stand for "X is a limit ordinal".

2. Uniform denumeration

Definition 2.1. $f: \omega_1 \rightarrow \omega_1^\omega$ is a *uniform denumeration* of ω_1 if for every α, $0 < \alpha < \omega_1$, $f(\alpha)$ is a function from ω onto α. Let U.D. be the assertion that there exists a uniform denumeration of ω_1.

Note. if there exists a uniform denumeration f, then there exists a uniform denumeration f^* such that for $\omega_0 \leqq \alpha < \omega_1$, $f^*(\alpha)$ is a one-one function from ω onto α.

Definition 2.2. $g: \omega_1 \rightarrow \omega_1^\omega$. The function g is a *uniform accessing function* on ω_1 if for every limit ordinal α with $\alpha < \omega_1$, $g(\alpha)$ is an ascending function from ω into α and the range of $g(\alpha)$ is cofinal in α. Let U.A. be the assertion that there exists a uniform accessing function on ω_1.

Definition 2.3. Given sets $A, B \subset \omega_1$, A is the *derivative* of B (written $A = B'$) if:

$$(Lm(x) \wedge (\exists^x t)\, t \in B) \leftrightarrow x \in A.$$

Definition 2.4. Let C.D. be the assertion that every closed set in ω_1 which contains only limit ordinals is a derivative. Note that C.D. can be stated in the monadic language.

We shall see that U.D., U.A., and C.D. are equivalent under ZF. (Note that ZFC implies all of them; also if in the definitions we replace ω_1 by β, $\beta < \omega_1$, then ZF implies U.D. (β), U.A. (β) and C.D. (β)).

DEFINITION 2.5. Let y be a set cofinal in ω_1 and $\alpha < \omega_1$, define the successor of $\Pi \subset \omega_1$, in y, α^+_y, by: $\alpha^+_y = \min(y - \alpha')$.

LEMMA 2.6. *There exists a set $\Pi \subset \omega_1$, such that Π is closed, it contains only limit ordinals and for every limit ordinal $\alpha < \omega_1$, there exists $\beta < \omega_1$ such that the interval $[\beta, \beta^+_\Pi)$ is isomorphic to α.*

PROOF. We define: $M = \{\langle x, y\rangle, y < x < \omega_1\}$ with the lexicographic order. M is isomorphic to ω_1, thus we can treat M as if it actually were ω_1.

We define: $\Pi = \{\langle x, 0\rangle \mid x$ is a limit ordinal or successor to a limit ordinal$\}$. Π contains only limit ordinals, Π is closed and every limit ordinal $\alpha < \omega_1$ is isomorphic to $[\langle \alpha, 0\rangle, \langle \alpha', 0\rangle)$ where $\langle \alpha', 0\rangle$ is the successor of $\langle \alpha, 0\rangle$ in Π.

Q.E.D.

THEOREM 2.7. *The following three are equivalent under* ZF:

1) U.D.

2) U.A.

3) C.D.

PROOF.

1) U.D. $\to$ U.A. From each function from ω onto a limit ordinal α we can obtain uniformly an ω sequence converging to α.

2). U.A. $\to$ C.D. Let Y be a set as above. Based on the assumption of uniform accessing we can obtain uniformly for every interval $[y, y^+_Y)$ with $y \in Y \cup \{0\}$ a subset Z_y of type ω cofinal in the interval. $Z = \bigcup_{y \in Y} Z_y \cup Z_0$ satisfies the equality $Z' = Y$.

3). U.A. $\to$ U.D. Let g be a u.a.f. (uniform accessing function). We shall construct a u.d. (uniform denumeration) f by induction on α as follows:

$$f(\alpha')(i) = \begin{cases} \alpha & \text{for} \quad i = 0 \\ f(\alpha)(i-1) & \text{for} \quad i \neq 0. \end{cases}$$

For α a lim. ord. (limit ordinal) we write α_i for $g(\alpha)(i)$. α_i is an ω sequence ascending to α. With the functions $f(\alpha_0)$ and $f(\alpha_{i+1} - \alpha_i)$ we obtain a mapping from $\omega \times \omega$ onto α. From this we obtain the function $f(\alpha)$ from ω onto α.

4) C.D. $\to$ U.A. Let Π be a set proved to exist in Lemma 2.6 and T such that $T' = \Pi$. For every lim. ord. α let β be the minimal ordinal for which $\alpha \approx [\beta, \beta^+_\Pi)$.

The set $T_\alpha = T \wedge [\beta, \beta^+_{\Pi})$ is cofinal of type ω on an interval isomorphic to α. From the T_α we construct the u.a.f.

LEMMA. 2.8. *If ω_1 is singular then* $\neg$ U.D.

PROOF. Let t_i be an ω sequence ascending to ω_1. If we assume that there exists a u.d.f. then using the functions $f(t_0)$ and $f(t_{i+1} - t_i)$ we obtain a mapping from $\omega \times \omega$ onto ω_1. Contradiction!

CONCLUSION 2.9. *$MT[\omega_1, <]$ is categorical and has finite axiomatization. That is: There is a monadic sentence L such that for every model $[A, <]$, $[A,<] \models L$ iff $[A, <]$ is isomorphic to $[\omega_1, <]$.*

L is the conjunction of the following:

1) *$<$ is a well-ordering.*
2) *There is no last element.*
3) ($\neg$ C.D.)$\vee$(*there is no cofinal set of type ω*).
4) *for every x,* (3) *does not hold relative to the interval* $[0, x)$.

3. Finite automata—definition

Let Σ be a finite set which we call the alphabet.

DEFINITION 3.1. A finite automaton is a triplet $\Omega = \langle S, N, L \rangle$ where: S is a finite set (the set of states), $N \subset S \times \Sigma \times S$, and $L \subset P(S) \times S$.

Input X is a function $X\colon \alpha \to \Sigma$ where $\alpha \leqq \omega_1$. α is the length of X and we write this $\alpha = l(X)$.

Let f be a function from an ordinal α into a finite set Q. For $x \leqq \alpha$ a limit ordinal we write:

$$\sup^x f = \{a \mid (\exists^x t) f(t) = a\}.$$

DEFINITION 3.2. Ω an automaton, X input of length $\alpha \leqq \omega_1$. $r\colon\alpha' \to S$ is a run of Ω on X if:

1) For every $t \leqq \alpha$, $\langle r(t), X(t), r(t') \rangle \in N$.
2) For every lim: ord $x \leqq \alpha$ $\langle \sup^x r, r(x) \rangle \in L$.

NOTE. The length of the run is one more than the length of the input.

An automaton is said to be *deterministic* if N is a function from $S \times \Sigma$ into S and L a function from $P(S)$ into S.

An automaton is said to be *free* if its alphabet consists of one letter, i.e., if its behavior is independent of the input. An accepting condition of an automaton is a criterion to decide if a particular run is a "good" run. For the time being let an accepting condition be a pair of states $\langle s_0, s_\infty \rangle$, a run r on input X will be a "good" one, if $r(0) = s_0$, $r(l(X)) = s_\infty$. (Later on we shall define another accepting condition for inputs of length ω_1.)

Definition 3.3. Let Ω be an automaton with accepting condition; then Ω "accepts" the input X if Ω has a "good" run on X.

Definition 3.4. Let g be a function; define
the range of g, $R(g) = \{x / \exists y\, g(y) = x\}$;
the domain of g, $D(g) = \{y \mid g \text{ is defined in } y\}$.

Let f be a function defined on an ordinal α. Let $[\beta, \gamma)$ be a sub-interval of α. Define $f[\beta, \gamma)$ to be the restriction of f to the interval $[\beta, \gamma)$.

Let X be an input of length $\alpha \leqq \omega_1$, $[\beta, \gamma)$ a subinterval of α. r is a run on $X[\beta, \gamma)$ if $r\colon [\beta, \gamma) \to S$ (we write, with abuse of the notation, $r[\beta, \gamma]$ to denote the domain of r) and r satisfies Definition 3.2 within the interval on which it is defined.

Definition 3.5. Let $r[\beta, \gamma)$ be a run on $X[\beta, \gamma)$; define the character of r, $C(r) \equiv \langle s_1, S', s_2 \rangle$ such that $s_1 = r(\beta)$, $s_2 = r(\gamma)$ and $S' = R(r[\beta, \gamma])$.

Definition 3.6. Let $[\beta, \gamma)$ be a countable subinterval of α $(\gamma < \omega_1)$; then the character of X in the interval $[\beta, \gamma)$ is

$$C(X[\beta, \gamma)) \equiv \{C(r[\beta, \gamma)) \mid r[\beta, \gamma) \text{ is a run of } \Omega \text{ on } X[\beta, \gamma)\}.$$

If no confusion will arise, we shall write $C([\beta, \gamma)) \equiv C(X[\beta, \gamma))$, and call it the character of the interval $[\beta, \gamma)$. Note that while the character of a run is a triple, the character of an input is a set of triples, the set of the characters of all the possible runs.

The character of a countable input contains all the information that the automaton Ω can gather from the input, that is if $C(X_1) = C(X_1')$, $C(X_2) = C(X_2')$ then $C(X_1 + X_2) = C(X_1' + X_2')$ (where $X + Y$ is the input X followed by the input Y) and therefore Ω accepts $X_1 + X_2$ iff Ω accepts $X_1' + X_2'$.

This will hold even for countable concatenation (if ω_1 is regular); the lemma of choice will be employed (see 5.2).

Now for uncountable concatenation U.D. is essential. Assume U.D. holds. Let $X = \Sigma_{i<\omega_1} X_i$, $X' = \Sigma_{i<\omega_1} X_i'$, $C(X_i) = C(X_i')$, $\forall i < \omega_1$ (and each X_i, X_i' is countable); it follows that for every run r of Ω on X there is a run r' of Ω on X', and a cofinal closed subset of ω_1, F, such that r and r' coincide on F.

The proof of the last remark is not trivial. It can be proved by the same methods used in Section 6.

4. Automata and the monadic language

Let's see the connection between automata and the monadic language. Every vector of subsets of α $\bar{Y} = \{Y_1, \cdots, Y_n\}$ is codable as input $\bar{X}$ in the following way: $\Sigma = \{0, 1\}^n$,

$$\bar{X}(t)(i) = \begin{cases} 1 & t \in Y_i \\ 0 & t \notin Y_i . \end{cases}$$

And on the other hand for every finite alphabet, Σ we may assume that $\Sigma \subset \{0, 1\}^n$ for a finite n. All input X can then be represented by a vector of sets as above.

From now on we shall assume that each input X of length α is a vector of subsets of α.

Now a run (a function from an ordinal to a finite set) is also codable as a vector of sets. For a given automaton Ω the assertion "r is a run of Ω on X" can be stated in the monadic language as a formula $\psi_\Omega(X, R)$ where R is the code for r.

The assertion "Ω accepts the input X" can be stated in the monadic language by a formula $A_\Omega(X)$ such that for every ordinal α and input X of length α:

$$[\alpha, <] \models A_\Omega(X) \text{ iff } \Omega \text{ accepts } X.$$

Moreover, the following lemma states that within the theory of well order, every MT-formula is generated from a formula of the form $A_\Omega(X)$ by quantification.

LEMMA 4.1. (Buchi.) (Prenex form for well order.) *Let $\Sigma(\bar{X})$ be an MT-formula in which the only free variables are the set variables $(X_1, X_2, \cdots, X_n) = \bar{X}$; then one can effectively find an automaton Ω and a formula $B(\bar{X}) \equiv$* (prefix in $\bar{Y}$) *$A_\Omega(\bar{Y}, \bar{X})$, where $\bar{Y}$ is a vector of set variables each occurring in the prefix, such that $\forall \bar{X}(\Sigma(\bar{X}) \leftrightarrow B(\bar{X}))$ holds in every well ordered model* (see [1, lemma 1.5]).

Now if for a given well ordered model $[\alpha, <]$ we can effectively eliminate quantifiers in formulas of the form (prefix in $\bar{Y}$)$A_\Omega(\bar{Y}, \bar{X})$ (regarding the formulas A_Ω as atomic formulas) and if for an input-free automaton Ω we can decide whether Ω accepts its (only) input then we have a decision method for $MT[\alpha, <]$.

5. The choice argument

Let an automaton Ω and an input X of length $\alpha \leqq \omega_1$ be given. Suppose α is divided into disjoint intervals $[\beta_i, \beta_{i'})$ and for each interval there is a run of Ω

defined in that interval satisfying a certain condition, such that if we can choose a unique run for each interval then they will all fit up to a run on α. It comes out that we don't need A.C. for countable choice, and we need only U.D. for uncountable choice.

LEMMA 5.1. (The lemma of choice.) *Let Ω be an automaton, $s_1, s_2 \in S$, $S_1 \subset S$; then there is a choice function H, $H: \Sigma^{<\omega_1} \times \omega_1^{\omega} \to S^{<\omega_1}$ such that if X is an input of length $\alpha < \omega_1$ and f is a mapping from ω_0 onto α, and there exists a run r of Ω on X whose character is $\langle s_1, S_1, s_2\rangle$, then $H(X, f)$ is such a run.*

PROOF. Assume S and $P(S)$ ordered. Let us order lexicographically the finite sequences in ω_1. Whenever we say "minimal" we mean with respect to the ordering above.

Let X be an input of length α such that there exists the required run on X, and let f be a denumeration of α. From f we obtain (uniformly) a sequence $\{t_i\}\ i < \omega_0$, such that every ordinal $t \leqq \alpha$ appears in the sequence infinitely often.

We shall build the run $H(X, f)$ by induction. Set $Q_{-1} = \{r \mid r$ is a run of Ω on X and $C(r) = \langle s_1, S_1, s_2\rangle\}$. Let $S_1 = (\bar{s}_1, \cdots, \bar{s}_n)$ and let $(\beta_1, \cdots, \beta_n) \subset \alpha'$ be the minimal sequence such that there exists a run $r \in Q_{-1}$ satisfying $r(\beta_J) = \bar{s}_J$, $\forall J \leqq n$. Set

$$Q = \{r \mid r \in Q_{-1} \text{ and } r(\beta_J) = \bar{s}_J, \forall J \leqq n\}.$$

By induction up to ω_0, using $\{t_i\}$ we build the following: Q_i, h_i, r_i. We start at $h_0 = \phi$, $r_0 = \phi$ and $Q_0 = Q$. r_i will be a function from a finite subset of α' into S (r_i will grow to be the chosen run r).

h_1 will be a function from a finite subset of α' to $P(S)$. h_i will be defined on limit ordinals on which r_i has already been defined. h_i will denote the sup of the run r at a limit ordinal.

Q_i will be the set of "good" runs on X.

The following will hold:

A) $r_i \subset r_{i+1}$, $h_i \subset h_{i+1}$ and $\phi \neq Q_{i+1} \subset Q_i$.

B) For every $r \in Q_i$, $r_i \subset r$.

C) $\langle t, S_1\rangle \in h_i \Rightarrow$ for each $r \in Q_i$: $\sup^t r = S_1$ and there exists $\tau < t$ such that for every $r \in Q_i$, $R(r[\tau, t)) = S_1$.

The process of induction continues thus: Let us assume r_i, h_i, Q_i are defined; we distinguish between the following cases:

1) r_i is not defined on t_i: Let s' be the minimal state such that there exists $r \in Q_i$ satisfying $r(t_i) = s'$. We define $r_{i+1} = r_i \cup \langle t_i, s'\rangle$. $h_{i+1} = h_i$ and $Q_{i+1} = \{r \mid r \in Q_i,\ r(t_i) = s'\}$.

2) r_i is defined on t_i, and t_i is not a limit ordinal. Pass to the next step, do not change r_i, h_i, Q_i.

3) r_i is defined on t_i, t_i is a limit ordinal, and h_i is not defined on t_i.

Let S' be the minimal subset of S such that there exists in Q_i a run r satisfying $\sup^{t_i} r = S'$ and let β be the minimal ordinal such that there exists a run r in Q_i satisfying, in addition to $\sup^{t_i} r = S'$, also $R(r[\beta, t_i)) = S'$. Define $r_{i+1} = r_i$, $h_{i+1} = h_i \cup \langle t_i, S' \rangle$ and

$$Q_{i+1} = \{r \mid r \in Q_i \wedge \sup^{t_i} r = S' \wedge R(r[\beta, t_i)) = S'\}.$$

4) r_i and h_i are defined on t_i, t_i a limit ordinal. Let $h_i(t_i) = S' = (s'_1, \cdots, s'_n)$ and $\gamma = \max(D(r_i) \wedge t_i)$. Let $(\beta_1, \cdots, \beta_n) \subset [\gamma', t_i)$ be the minimal sequence such that there exists $r \in Q_i$ satisfying $r(\beta_J) = s'_J \forall J \leqq n$. (Such an r exists because every $r \in Q_i$ satisfies $\sup^{t_i} r = S'$.)

Define: $r_{i+1} = r_i \cup \{\langle \beta_J, s'_J \rangle\}_{J \leqq n}$, $h_{i+1} = h_i$ and

$$Q_{i+1} = \{r \mid r \in Q_i \text{ and } \forall J \leqq n, r(\beta_J) = s'_J\}.$$

Now define $\bar{r} = \bigcup_{i<\omega} r_i$; let us verify that $\bar{r}$ is a run of Ω on X. $\bar{r}$ is defined for all t, $t \leqq \alpha$ (case 1). For $t < \alpha$ let i be the first such that r_i is defined at t and t'. Now for every $r \in Q_{i+1}$, $r(t) = \bar{r}(t)$, $r(t') = \bar{r}(t)$ and Q_{i+1} is not empty, therefore $\langle \bar{r}(t), X(t), \bar{r}(t') \rangle \in N$ holds (see Definition 3.2).

Let x be a limit ordinal, $x \leqq \alpha$. At the second time x appeared in $\{t_i\}$ we defined (case 3) $h_i(x) = S'$ and set a $\beta < x$ such that for all $r \in Q_{i+1}$, $R(r[\beta, x)) = S'$, thus forcing $\sup^x \bar{r} \subset S'$. Now for every $t < x$, at the first time we encountered x after r_i has been defined at t, we did choose, for every $s' \in S'$, $t < \beta' < x$ and determined $\bar{r}(\beta') = s'$. Therefore, $\sup^x \bar{r} = S'$.

At the same time we have set $h_i(x) = S'$, there was a run r in Q_i satisfying $\sup^x r = S'$ and $r(x) = r_i(x)$, hence $\langle \sup^x \bar{r}, \bar{r}(x) \rangle \in L$. So $\bar{r}$ is a run of Ω on X. Let us see that $\bar{r}$ has the character $\langle s_1, S_1, s_2 \rangle$. Every $r \in Q_0$ satisfies $r(0) = s_1$, $r(\alpha) = s_2$, therefore $\bar{r}$ satisfies these conditions. Surely $R(\bar{r}) \subset S_1$; now for every $s' \in S_1$ we determine $\beta \leqq \alpha$ such that $r(\beta) = s'$ for every $r \in Q_0$, therefore $\bar{r}(\beta) = s'$ and $R(\bar{r}) = S_1$. Set $H(X, f) = \bar{r}$. Q.E.D.

LEMMA 5.2. (Countable splicing.) *Let Ω be an automaton, X input of length α, $\alpha < \omega_1$, α a limit ordinal, s', $s^* \in S$, $D \subset S$, $\langle D, s^* \rangle \in L$ and $\{\beta_i\}$ an ω sequence ascending to α such that*:

1) *There exists a run $r[0, \beta_0]$ on $X[0, \beta_0)$ with $r(\beta_0) = s'$.*

2) *For each i there exists a run $r[\beta_i, \beta_{i+1}]$ on $X[\beta_i, \beta_{i+1})$ satisfying $C(r[\beta_i, \beta_{i+1}]) = \langle s', D, s' \rangle$.*

Then there exists a run r on X such that $r[0,\beta_0]$ *satisfies* (1) *and* $r[\beta_i,\beta_{i+1}]$ *satisfies* (2) *for each i, and* $r(\alpha)=s^*$.

PROOF. Let f be a denumeration of α. From f we obtain $\{f_i\}$ where f_i is a denumeration of the interval $[\beta_i,\beta_{i+1})$. By the choice lemma we can choose runs $r[\beta_i,\beta_{i+1}]$ and splice them together, then add a proper run $r[0,\beta_0)$, and set $r(\alpha)=s^*$. Q.E.D.

In Buchi's work, the use of A.C. for deciding procedure for countable ordinals is isolated to the countable splicing argument. Therefore, all Buchi's results concerning countable ordinals hold under ZF.

LEMMA 5.3. (Uncountable splicing.) *Assume* U.D. *Let* Ω *be an automaton, X input of length* ω_1, $G\subset Q\subset S$, *Y a cofinal closed subset of* ω_1. $Y=\bigcup_{s\in G}Y_s$ *where* Y_s *are disjoint (for* $y\in Y$ *we denote by* s_y *the state s satisfying* $y\in Y_s$*) such that*:

1) $\langle Q,s\rangle\in L$ *for all* $s\in G$.

2) *Denote* $y_0=\min Y$. *There exists a run r on* $[0,y_0)$ *satisfying* $r(0)=s_0$, $r(y_0)=s_{y_0}$.

3) *Let* y^+ *denote* $\min(Y-y')$ *(i.e., the successor of y in Y). For each* $y\in Y$ *there exists a run r on* $[y,y^+)$ *satisfying* $C(r[y,y^+])=\langle s_y,Q,s_{y^+}\rangle$.

Under these assumptions there exists a run r on all X such that $r[0,y_0]$ *satisfies* (2) *and for each* $y\in Y$, $r[y,y^+]$ *satisfies* (3).

PROOF. Given a uniform denumeration, by the choice lemma we can choose for each interval $[y,y^+)$ and $[0,y_0)$ an appropriate run. It follows from (1) and (3) that they all fit together, and as Y is closed and cofinal, they cover all of ω_1.

Q.E.D.

It follows that the uncountable splicing property is equivalent to U.D. Let us assume that ω_1 satisfies the uncountable splicing; then we shall see that every cofinal closed set of limit ordinals is a derivative. Let X be such a set. We define an automaton Ω which, while running on the input X, "tries to guess" a set B such that $X=B'$.

$$\Sigma=\{0,1\},\ S=\{a,b,c,z\},\ \text{the initial state } s_0=\alpha.$$

N is defined according to the following scheme:

$$c\overset{0}{\to}z;\quad c\overset{1}{\to}a,b;\quad b\overset{0}{\to}a,b;\quad a\overset{0}{\to}a,b;\quad a,b\overset{1}{\to}z;\quad z\overset{0,1}{\to}z.$$

L is even deterministic and satisfies:

$$L(\{z\}) = z$$

$$L(A) = c \text{ if } b \in A$$

$$L(A) = a \text{ otherwise.}$$

Note that if the automaton moves into state z, it will stay in that state forever. Now if r is a run of Ω on X such that $z \notin R(r)$ then $X = r^{-1}(c)$, and $B = r^{-1}(b)$ satisfies $B' = X$. Now for a cofinal closed X of limit ordinal, the assumption of the uncountable splicing theorem holds for $Q = \{a, b, c\}$, $G = \{c\}$, $Y_c = Y = X'$ and $s_0 = a$ (as every closed bounded set of limit ordinals is a derivative); therefore there exists a run r which avoids z.

6. Automata on uncountable input

Let F be the filter generated in ω_1 by the cofinal closed sets (i.e., $Y \in F$ if Y has a subset Z, Z is closed and cofinal in ω_1).

We say that $Z \subset \omega_1$ is null if $(\omega_1 - Z) \in F$. Note that $Y \subset \omega_1$ is not null if Y intersects every cofinal closed set.

Let $\equiv$ be the equivalence relation between subsets of ω_1 modulo F (i.e. $X \equiv Y$ if $(X - Y) \cup (Y - X)$ is null). For $Y \subset \omega_1$ let $\bar{Y}$ denote the equivalence class of Y. Now A.C. forces the boolean algebra $P(\omega_1)/F$ to be an atomless boolean algebra (else ω_1 was measurable). In fact U.D. and "F is $\aleph_1$ complete" alone imply that each non null element of $P(\omega_1)/F$ contains $\aleph_1$ disjoint non null elements (see [1, theor. 5.6]). The structure of $P(\omega_1)/F$ without A.C. is not clear, and it appears that ZF, and even ZF + U.D. do not determine the first order theory of $P(\omega_1)/F$.

As every first order sentence in $P(\omega_1)/F$ can be stated in the monadic order language, we can not have a decision procedure which will be independent of $P(\omega_1)/F$. The best we can do is to translate every MT sentence into a first order sentence of $P(\omega_1)/F$. We will see that if U.D. holds, then there is such an effective translation.

A Buchi accepting condition for input of length ω_1 is a pair $\langle s_0, G\rangle$, $s_0 \in S$, $G \subset P(S)$. The automaton Ω accepts X if it has a run r satisfying $r(0) = s_0$, $\{s \mid \overline{r^{-1}(s)} \neq 0\} \in G$ ($r^{-1}(s)$ is the set of all ordinals t such that $r(t) = s$).

We shall use a finer accepting condition.

Definition 6.1. An accepting condition for inputs of length ω_1 will be a pair $\langle s_0, \psi\rangle$ where $s_0 \in S$ and ψ is a first order formula in the language of boolean algebra whose free variables are $\{R_s \mid s \in S\}$.

An automaton Ω accepts the input X, if it has a run r satisfying $r(0) = s_0$, and $P(\omega_1)/F \models \psi(r)$ *where* $\psi(r)$ is the formula ψ with the assignment $\bar{r}^{1}(s)$ for R_s. Note that in our terms, Buchi's accepting condition is a formula $\psi[R_s]$ which is a boolean form (has no quantification) in the atomic formulas $R_s \neq \phi$. On the other hand, if $P(\omega_1)/F$ is atomless, then its theory admits elimination of quantifiers, and since the R_s are disjoint and their union is 1, every formula ψ is equivalent to a boolean form in the atomic formulas $R_s \neq \phi$. That is, Buchi's accepting conditions are equivalent to ours if $P(\omega_1)/F$ is atomless.

Our aim is to find a quantifier elimination method for the formulas of the form (prefix in Y) $A_\Omega(Y, X)$, that is to show that the class of sets of input defined by an automaton is effectively closed under projection and complementation. (A set of inputs is defined by an automaton, if all its members are in the same alphabet and there is an automaton Ω such that the set contains exactly the inputs which Ω accepts.)

As our automata are nondeterministic, that class is trivially closed under projection. Also that class is closed under union and intersection. (To define the inputs accepted by both automata Ω, Ω', we will have to "run in parallel" the automata Ω and Ω'.) Our main goal is to show that that class is closed under complementation, i.e., given an automaton Ω, to effectively find an automaton $\bar{\Omega}$, in the same alphabet such that $\bar{\Omega}$ accepts exactly the inputs which Ω does not accept.

Buchi's fundamental result is that for countable inputs, deterministic automata are equivalent to nondeterministic ones, that is

THEOREM 6.2. (Buchi.) *For any automaton Ω, one can effectively find a deterministic automaton Ω^* such that for every $\alpha < \omega_1$ and every input X of length α, Ω accepts X iff Ω^* accepts X.* [(See [1, lemma 4.4].)

Using the fact that the behavior of an automaton on a countable input can be simulated by a deterministic one, Buchi captured the essence of an uncountable input in the following lemma.

LEMMA 6.3. (Buchi.) *Let an automaton Ω and an input X of length ω_1 be given, let C denote the set of all formally possible input characters (i.e., $C = P(S \times P(S) \times S)$); then there is an input character e, and $\{P_c\}$ $c \in C$ subsets of ω_1 such that:*

1) *$P_c \wedge P_{c'} = \phi$ for $c \neq c'$.*
2) *Let $P = \bigcup_{c \in C} P_c$; P is a cofinal closed set.*
3) *For each $\alpha \in P$, $C([0, \alpha)) = e$.*
4) *If $\alpha \in P_c$, $\beta \in P$, $\alpha < \beta$, then $C[\alpha, \beta) = c$.*

(NOTE. *Many of the P_c may be empty.) The character e is unique, each P_c is unique modulo the filter F.* (See [1, lemma 6.2].)

Let us call the finite sequences $\langle e, \{\bar{P}c\}\rangle c \in C$ the essence of the input X.

The assertion "$\langle e, \{P_c\}\rangle$ is the essence of the input X" can be verified by an automaton in the following sense.

LEMMA 6.4. (Buchi.) *Let Ω be an automaton; for each $e \in C$ one can effectively find an automaton with accepting condition Ω_e whose input has two components, one an original input of Ω and the other a vector of $\| C \|$ sets which we denote by $\{P_c\}$, such that for inputs of length ω_1: Ωe accepts $X \cdot \{P_c\}$ iff e and $\{P_c\}$ satisfied* (1) *through* (4) *of Lemma* 6.3. (See [1, lemma 6.4].)

This automaton Ω_e may be deterministic; the main idea is to run simultaneously several copies of a deterministic automaton which stimulates the behavior of Ω. At any time $t \in \cup P_c$, a copy is employed. Two copies employed for the same set P_c, may be merged if they are in the same state, a copy which is employed for the set $P_{c'}$ should yield, at any time $t \in \cup P_c$, that the character of the input interval it has examined is c', else the input is rejected.

From now on we assume U.D.

LEMMA 6.5. *Let Ω be an automaton with accepting condition $\langle s_0, \psi\rangle$. Let X be an ω_1 input whose essence is $\langle e, \{\bar{P}_c\}\rangle$; then Ω accepts X iff there are G, Q, $G \subset Q \subset S$, and disjoint elements of $P(\omega_1)/F, \{\bar{P}_{c,s} \mid c \in C, s \in G\}$, such that:*

1) *$\langle Q, s\rangle \in L$ for each $s \in G$.*
2) *For each $s \in G$ there is $H \subset S$ such that $\langle s_0, H, s\rangle \in e$.*
3) *$\bar{P}_{c,s} \neq \phi$ implies that for each $s' \in G$, $\langle s, Q, s'\rangle \in c$.*
4) *$\bar{P}_c = \bigcup_{s \in G} \bar{P}_{c,s}$.*
5) *Set $\bar{P}_s = \bigcup_{c \in C} \bar{P}_{c,s}$ for $s \in G$ and $\bar{P}_s = \phi$ for $s \notin G$.*

Then $\psi(\{\bar{P}_s\})$ holds in $P(\omega_1)/F$.

PROOF. First let us choose representatives $\{P_c\}$ for $\{\bar{P}_c\}$ such that (1) through (4) of Lemma 6.3 hold.

Assume Ω accepts X. Let r be a good run of Ω on X. Set $G = \{s \mid r^{-1}(s) \neq \phi\}$, $Q = \sup^{\omega_1} r$, $\bar{P}_{c,s} = \bar{P}_c \wedge \overline{r^{-1}(s)}$. Let γ be such that $R(r[\gamma, \omega_1)) = Q$; set $F = \{x \mid \sup^x r = Q, x > \gamma\}$. F is cofinal closed and therefore intersects each $r^{-1}(s)$ for $s \in G$; thus we have (1). As $\cup P_c$ is closed cofinal and, by (3) of Lemma 6.3, (2) holds too. For (3) if $\bar{P}_{c,s} \neq \phi$ then $P_c \wedge r^{-1}(s) \wedge [\gamma, \omega_1) \neq \phi$. Let α belong to that set. Now $[\alpha', \omega_1) \wedge F \wedge (\cup P_c)$ is closed cofinal and intersects each $r^{-1}(s')$, $s' \in G$; $\langle s, Q, s'\rangle \in c$ by (4) of Lemma 6.3. (4) holds because $\bigcup_{s \in G} \overline{r^{-1}}(s) = 1$.

Now as $\cup \bar{P}c = 1$ we have $\bar{P}_s = \overline{r^{-1}}(s)$, r is a good run; therefore (5) follows.
Q.E.D. one side.

Assume the other side. Choose representatives $P_{c,s}$ for the $\bar{P}_{c,s}$. We may assume that the $P_{c,s}$ are disjoint and that assertion (3) and (4) hold for the $P_{c,s}$ and the P_c (without $\bar{}$); otherwise we can intersect them all by a cofinal closed set.

Set $P_s = \bigcup_c P_{c,s}$. Now the condition of Lemma 5.3 (lemma for uncountable splicing) holds when we replace Y_s by P_s and Y by $\bigcup_{s \in G} P_s = \bigcup_c P_c$. (Let $y < x$, $y \in P_{s_1}$ $x \in P_{s_2}$; there is c' such that $y \in P_{c'}$. As $P_{c'} \wedge P_{s_1} \neq \phi$, by (3) $\langle s_1, Q, s_2 \rangle \in c'$. Now $x \in \bigcup Pc$ and by (4) of Lemma 6.3, $C[y, x) = c'$, therefore there is a run $r[y, x)$ satisfying $C(r[y, x)) = \langle s_1, Q, s_2 \rangle$.)

By Lemma 5.3, there is a run r of Ω on X such that $r(0) = s_0$ and $r(y) = s$ for $y \in P_s$. Now $\bigcup_{s \in G} P_s$ is cofinal closed; hence $r^{-1}(s) \equiv P_s$ for $s \in G$ and $r^{-1}(s) \equiv \phi$ for $s \notin G$. By (5) r is a good run. That is, Ω accepts X. Q.E.D.

CONCLUSION 6.6. *Let Ω be an automaton $e \in C$. Then one can effectively find a formula ψ_e in the first order language of boolean algebra, whose only free variables are $\{P_c\}$, $c \in C$, such that for every input X whose essence is $\langle e, \{\bar{P}_c\}\rangle$, Ω accepts X iff $\psi_e(\{\bar{P}_c\})$ holds in $P(\omega_1)/F$.*

LEMMA 6.7. *Let $\phi(Y)$ be a boolean algebra formula in the variables $Y = \langle Y_1, \cdots, Y_n \rangle$. One can effectively find an automaton Ω with accepting condition, whose input is a vector Y of n sets such that: Ω accepts Y iff $\phi(\bar{Y}_1, \cdots, \bar{Y}_n)$ holds in $P(\omega_1)/F$.*

PROOF. Trivial.

Now let us join all our lemmas into the complementation proof.

LEMMA 6.8. *Let Ω be an automaton with accepting condition; then one can effectively find an automaton $\bar{\Omega}$ on the same alphabet such that Ω accepts X iff $\bar{\Omega}$ does not accept X.*

PROOF. We shall use the fact that our automata are closed under union, intersection, projection and cylindrification (the proof is left to the reader). Let's start with inputs of the form $X \cdot \{P_c\}$ where X is an original input and $\{P_c\}$, $c \in C$, is a vector of $\| C \|$ sets.

Let $e \in C$. By Lemma 6.4 there is an automaton $\Omega_{e,1}$ such that: $\Omega_{e,1}$ accepts $X \cdot \{P_c\}$ iff $\langle e, \bar{P}_c \rangle$ is the essence of X and the P_c are good representatives of the $\bar{P}_c$ (i.e., the P_c satisfies (1) through (4) of Lemma 6.3)

By Lemma 6.7 there is an automaton $\Omega_{e,2}$ such that $\Omega_{e,2}$ accepts $X \cdot \{P_c\}$ iff $\neg\, \psi_e(\{\bar{P}c\})$ holds in $P(\omega_1)/F$. (ψ_e is the formula of Conclusion 6.6.) Let $\Omega_{e,3}$ be the intersection of $\Omega_{e,1}$ and $\Omega_{e,2}$, that is:

$\Omega_{e,3}$ accepts $X \cdot \{P_c\}$ iff $\langle e, \{\bar{P}_c\}\rangle$ is the essence of X, P_c are good representatives and $\neg\, \psi_e(\{\bar{P}c\})$ holds in $P(\omega_1)/F$. By the definition of ψ_e we get that $\Omega_{e,3}$ accepts $X \cdot \{P_c\}$ iff $\langle e, \{P_c\}\rangle$ is the essence of X, P_c are good representatives and Ω doesn't accept X.

Now let $\Omega_{e,4}$ be the projection of $\Omega_{e,3}$ on the X component. That is, $\Omega_{e,4}$ runs on original inputs and $\Omega_{e,4}$ accepts X iff e is in the essence of X and Ω doesn't accept X.

Let $\bar{\Omega}$ be the union of the automata $\Omega_{e,4}$ for all $e \in C$. Then $\bar{\Omega}$ accepts X iff Ω doesn't accept X. Q.E.D.

Now we arrive to the input free case.

LEMMA 6.9. (Buchi.) *Let Ω be an input free automaton. Then the essence of its only input is in the format $\langle e, \{\bar{P}_c\}\rangle$ where $\bar{P}_c = \phi$ for $c \neq e$ and $\bar{P}_e = 1$. Moreover e can be effectively found.* (See [1, remark 6.8].)

LEMMA 6.10 *Let Ω be an input free automaton with accepting condition. Then one can effectively construct a boolean algebra sentence ψ_Ω such that: Ω accepts its only input iff ψ_Ω holds in $P(\omega_1)/F$.*

PROOF. Use the formula $\psi_e(\{\bar{P}_c\})$ of Conclusion 6.7. Substitute 1 for $\bar{P}_e$ and 0 for $\bar{P}_c$, $c \neq e$.

THEOREM 6.11. *$MT[\omega_1, <]$ is recursive in $T[P(\omega_1)/F, \cup, \wedge, -]$. Any monadic sentence Σ can be effectively translated into a boolean algebra sentence ψ such that: Σ holds in $[\omega_1, <]$ iff ψ holds in $P(\omega_1)/F$.*

PROOF. Given a monadic sentence Σ, first put it in a prenex form (prefix in X) $A_\Omega(X)$. (See Lemma 4.1.) Then eliminate quantifiers by projection and complementation until you reach the input free case. By Lemma 6.10, ψ can be effectively constructed. Q.E.D.

7. Complete theories of boolean algebras

Let B be a boolean algebra. An element b of B is called atomic if for every d, $0 \neq d \subset b$, there is an atom e, $e \subset d$. An element b is called atomless if it contains no atom.

Note that 0 is both atomic and atomless. For the algebra B define the ideal I, $I = \{a \cup b \mid a$ is atomic and b is atomless$\}$. Define $B' = B/I$.

Now define by induction:

$$B^0 = B$$

$$B^{i+1} = (B^i)'.$$

We say that an algebra is trivial if $1 = 0$. Now following Chang and Keisler (3), we assign to each boolean algebra an invariant $\langle m(B), n(B)\rangle$ as follows:

$$m(B) = \begin{cases} \text{the least } k < \omega \text{ such that } B^k \text{ is trivial, if such a } k \text{ exists} \\ \infty, \text{ otherwise} \end{cases}$$

$$n_0(B) = \begin{cases} \infty: m(B) = k, 0 < k < \infty \text{ and } B^{k-1} \text{ has infinitely many atoms} \\ j: m(B) = k, 0 < k < \infty \text{ and } B^{k-1} \text{ has exactly } j < \omega \text{ atoms} \end{cases}$$

$$n(B) = \begin{cases} 0: m(B) = \infty \\ n_0(B): m(B) = k < \omega \text{ and } B^k \text{ is atomic} \\ -n_0(B): m(B) = k < \omega \text{ and } B^k \text{ is not atomic.} \end{cases}$$

Now the invariant can be expressed in the first order language, finite invariants by a sentence and infinite invariants (which contain ∞) by an infinite set of sentences.

It turns out that the invariant determines the first order theory, and more than that:

THEOREM 7.1. (Tarski.) *Let T_0 be the theory of boolean algebra. Every complete extension of T_0 is recursive. Moreover, the decision procedure is uniform in the invariant of the complete theory.* (See [3].)

THEOREM 7.2. ZF $\vdash$ U.D. $\rightarrow$ "$MT[\omega_1, <]$ *is recursive*".

PROOF. We have proved within ZF + U.D. that $MT[\omega_1, <]$ is recursive in $T(P(\omega_1)/F)$. Tarski's assertion is proved within ZF. Q.E.D.

PROBLEM 1. Which of the complete boolean algebra theories are admissible as the theory of $P(\omega_1)/F$ under ZF + U.D.?

A related problem is

PROBLEM 2. Let $\overline{MT}[\omega_1, <]$ be the set of sentences which hold in $[\omega_1, <]$ for every model M of ZF + U.D. (that is, the set of sentences Σ such that ZF+U.D. $\vdash$ "$[\omega_1, <] \models \Sigma$"). Is $\overline{MT}[\omega_1, <]$ recursive?

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